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Invariant measures for equicontinuous semigroups of continuous transformations of a compact Hausdorff space

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Abstract

Equicontinuous semigroups of transformations of a compact Hausdorff space and their sets of all invariant (Borel, regular and probabilistic) measures are studied. Conditions equivalent to the existence of at least one invariant measure are given. The (algebraic and topological) structure of the set of invariant measures is researched.

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The classical Haar measure theorem (in the compact case) (see e.g. [5]) states that for any compact group G there exists a unique Borel, regular and probabilistic measure which is invariant under each mapping of the form $G \ni x \mapsto axb \in G$ (where $a, b \in G$). Such a measure enables to build the harmonic analysis on a compact group.

The aim of this paper is to generalize the 'compact' version of the Haar measure theorem. For this, we shall consider a completely arbitrary compact Hausdorff space X and any equicontinuous semigroup $\mathcal{G}$ of its continuous transformations and we shall study the set $\mathrm{Inv}(\mathcal{G})$ of all Borel, regular and probabilistic measures which are invariant under each mapping of that semigroup. The main results of the paper give the conditions equivalent to the nonemptiness of $\mathrm{Inv}(\mathcal{G})$ (Theorem 3.6, Propositions 3.9 and 3.10). It turns out that if $\mathrm{Inv}(\mathcal{G})$ is nonempty, then it is a Choquet symplex (Theorems 2.5 and 3.6). We will also give a condition for the existence of an invariant (i.e. left and simultaneously right invariant) measure on a compact semigroup (see Theorem 3.12).

There is a huge range of literature concerning the generalization of the Haar measure theorem for groups of transformations. See [8] for references. For more information about invariant measures on topological semigroups see [2].

1. Preliminaries

In this section we introduce all notions of this paper and cite well-known theorems which will be used in the next sections.

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Firstly some topological notions. If X and Y are topological spaces, the space of all continuous functions from X to Y is denoted by $\mathcal{C}(X, Y)$. The notation "cl A" denotes the closure of a set A.

Definition 1.1. *The compact-open topology of* $\mathcal{C}(X, Y)$ is the topology with the basis consisting of finite intersections of sets of the following form:

$$B(K, U) := \{f \in \mathcal{C}(X, Y) \colon\ f(K) \subset U\},$$

where $K \subset X$ is compact and $U \subset Y$ is open.

A family $\mathcal{F} \subset \mathcal{C}(X, Y)$ is *equicontinuous* if for any points $x \in X$, $y \in Y$ and every open neighborhood $V \subset Y$ of the point y there exist open subsets $U \subset X$, $W \subset Y$ such that $x \in U$, $y \in W$ and for each $f \in \mathcal{F}$, $f(U) \subset V$ provided $f(x) \in W$.

We shall always consider the space $\mathcal{C}(X, Y)$ *and its subsets with the above topology*. If X and Y are compact Hausdorff spaces, compact subsets of $\mathcal{C}(X, Y)$ are characterized as those closed subsets which are equicontinuous (the Ascoli type theorem; see [1, Theorem 3.4.20] for proof). If, in addition, Z is any topological space, the operation of composing functions, as a map between $\mathcal{C}(Y, Z) \times \mathcal{C}(X, Y)$ and $\mathcal{C}(X, Y)$, is continuous [1, Theorem 3.4.2].

If X is compact and Y is metric, the compact-open topology describes the uniform convergence, which, in consequence, does not depend on the choice of a metric on Y. Moreover, in this case the space $\mathcal{C}(X, Y)$ is metrizable and if $Y = \mathbb{R}^n$, the supremum norm (induced by the euclidean norm on $\mathbb{R}^n$) on $\mathcal{C}(X, \mathbb{R}^n)$ induces the compact-open topology.

In the next sections we will use the following result (for proof see [1, Theorem 3.4.3]).

Proposition 1.2. *If X is a compact Hausdorff space, Y and Z are any topological spaces and $V : Z \to \mathcal{C}(X, Y)$, the following conditions are equivalent*:

(i) *V is continuous,*
(ii) *the mapping $X \times Z \ni (x, z) \mapsto (V(z))(x) \in Y$ is continuous.*

The following result states that a *compact* group is the same as a compact semigroup which is algebraically a group.

Lemma 1.3. *If $\mathcal{G}$ is an algebraic group and simultaneously a compact Hausdorff space such that the multiplication is a continuous map between $\mathcal{G} \times \mathcal{G}$ and $\mathcal{G}$, then $\mathcal{G}$ is a topological group, i.e. the operation of taking inverses is continuous.*

Proof. Since $\mathcal{G}$ is compact, it is enough to prove that the graph $\{(x, x^{-1}) \colon\ x \in \mathcal{G}\}$ is closed. If $(x_\sigma)_{\sigma \in \Sigma}$ is such a net that $\lim_{\sigma \in \Sigma} x_\sigma = x$ and $\lim_{\sigma \in \Sigma} x_\sigma^{-1} = y$ (for some $x, y \in \mathcal{G}$), then—by the continuity of the multiplication—$x \cdot y = \lim_{\sigma \in \Sigma} x_\sigma \cdot x_\sigma^{-1} = e$, hence $y = x^{-1}$. □

Now we remind basic properties of spaces of measures. For simplicity, we will assume that X is a compact Hausdorff space. By $\mathrm{Prob}(X)$ we denote the set of all Borel, regular and probabilistic measures on X. *The set* $\mathrm{Prob}(X)$ *and all its subsets will be always considered with the classical weak topology*, i.e. the topology with the basis consisting of finite intersections of sets of the form:

$$B(\mu; f, \varepsilon) := \left\{\lambda \in \mathrm{Prob}(X) \colon\ \left|\int_X f\,\mathrm{d}\mu - \int_X f\,\mathrm{d}\lambda\right| < \varepsilon\right\},$$

where $\mu \in \mathrm{Prob}(X)$, $f \in \mathcal{C}(X, \mathbb{R})$ and $\varepsilon > 0$. The well-known fact states that $\mathrm{Prob}(X)$ is compact with respect to this topology (which follows from the Riesz type theorem characterizing continuous functionals of $\mathcal{C}(X, \mathbb{R})$ and the Banach–Alaoglu one). What is more, the function

$$\mathcal{C}(X, \mathbb{R}) \times \mathrm{Prob}(X) \ni (f, \mu) \longmapsto \int_X f\,\mathrm{d}\mu \in \mathbb{R} \tag{1.1}$$

is continuous. But for us the following fact is more important.

Lemma 1.4. *If X and Y are compact Hausdorff spaces, the function*

$$\mathcal{C}(X,Y)\times \mathrm{Prob}(X)\ni(\varphi,\mu)\longmapsto \mu\circ\varphi\in\mathrm{Prob}(Y)$$

is continuous, where $(\mu\circ\varphi)(B):=\mu(\varphi^{-1}(B))$ for a Borel subset B of Y.

Proof. Let $((\varphi_\sigma,\mu_\sigma))_{\sigma\in\Sigma}\subset\mathcal{C}(X,Y)\times\mathrm{Prob}(X)$ be any net convergent to some $(\varphi,\mu)\in\mathcal{C}(X,Y)\times\mathrm{Prob}(X)$. We have to show that the net $(\mu_\sigma\circ\varphi_\sigma)_{\sigma\in\Sigma}$ converges to $\mu\circ\varphi$. It holds if and only if $\lim_{\sigma\in\Sigma}\int_Y f\,\mathrm{d}(\mu_\sigma\circ\varphi_\sigma)=\int_Y f\,\mathrm{d}(\mu\circ\varphi)$ for any $f\in\mathcal{C}(Y,\mathbb{R})$. But $\int_Y g\,\mathrm{d}(\lambda\circ\psi)=\int_X(g\circ\psi)\,\mathrm{d}\lambda$ for any $g\in\mathcal{C}(Y,\mathbb{R})$, $\psi\in\mathcal{C}(X,Y)$ and $\lambda\in\mathrm{Prob}(X)$. Further, thanks to the continuity of the operation of composing functions, for $f\in\mathcal{C}(Y,\mathbb{R})$ we have $\lim_{\sigma\in\Sigma}(f\circ\varphi_\sigma)=f\circ\varphi$ and finally, by the continuity of the mapping (1.1), we obtain:

$$\lim_{\sigma\in\Sigma}\int_Y f\,\mathrm{d}(\mu_\sigma\circ\varphi_\sigma)=\lim_{\sigma\in\Sigma}\int_X(f\circ\varphi_\sigma)\,\mathrm{d}\mu_\sigma=\int_X(f\circ\varphi)\,\mathrm{d}\mu=\int_Y f\,\mathrm{d}(\mu\circ\varphi).\qquad\square$$

Definition 1.5. If $\mathcal{F}$ is any family of continuous transformations of a compact Hausdorff space X, then by $\mathrm{Inv}(\mathcal{F})$ we denote *the set of all Borel, regular and probabilistic measures which are invariant under each mapping of $\mathcal{F}$*. So

$$\mathrm{Inv}(\mathcal{F})=\{\mu\in\mathrm{Prob}(X)\mid\forall\varphi\in\mathcal{F}\colon\ \mu\circ\varphi=\mu\}.$$

An immediate consequence of Lemma 1.4 is the following

Corollary 1.6. *If $\mathcal{F}$ is a family of continuous transformations of a compact Hausdorff space X and $\mathcal{G}$ denotes the closure of the semigroup generated by $\mathcal{F}$, then $\mathcal{G}$ is a closed subsemigroup of the semigroup $(\mathcal{C}(X,X),\circ)$,* $\mathrm{Inv}(\mathcal{F})$ *is compact and convex and* $\mathrm{Inv}(\mathcal{F})=\mathrm{Inv}(\mathcal{G})$.

The above corollary says that it is enough to study compact semigroups of transformations instead of equicontinuous ones.

Proposition 1.7. *For any family $\mathcal{F}$ of continuous transformations of a* ***metrizable*** *compact space $X\neq\emptyset$ the following conditions are equivalent*:

(i) *there exists a metric d on X which induces the topology of X and such that for each $f\in\mathcal{F}$ the inequality $d(f(x),f(y))\leqslant d(x,y)$ holds for any $x,y\in X$ (for simplicity: f is a contraction with respect to d).*
(ii) *the semigroup generated by $\mathcal{F}$ is equicontinuous.*

Proof. (i) $\Rightarrow$ (ii): If $\mathcal{F}$ consists of contractions with respect to d, so is the semigroup generated by $\mathcal{F}$. So '(ii)' follows from the facts that the set of all contractions on a compact metric space is compact in the topology of uniform convergence and that this topology is identical with the compact-open one.

(ii) $\Rightarrow$ (i): Let $\mathcal{G}$ denote the closure of the semigroup generated by $\mathcal{F}$. It is a compact subset of $\mathcal{C}(X,X)$. Let ϱ be any metric on X which induces the topology of this space. Let

$$d\colon X\times X\ni(x,y)\longmapsto\sup_{\varphi\in\mathcal{G}\cup\{\mathrm{id}_X\}}\varrho\big(\varphi(x),\varphi(y)\big)\in[0,+\infty].\tag{1.2}$$

First of all, note that $d(x,y)\in\mathbb{R}$ for all $x,y\in X$. Indeed, the semigroup $\mathcal{G}$ is compact in the topology of uniform convergence (with respect to ϱ) and the function $\mathcal{G}\ni\varphi\mapsto\varrho(\varphi(x),\varphi(y))\in\mathbb{R}$ (with fixed $x,y\in X$) is continuous and so bounded. One can easily prove that d is a metric and $\mathcal{G}$ consists of contractions with respect to it. It suffices to show that d and ϱ induce the same topologies. On the one hand, since $\mathrm{id}_X\in\mathcal{G}\cup\{\mathrm{id}_X\}$, $\varrho\leqslant d$. But on the other hand, $\mathcal{G}$ is uniformly equicontinuous with respect to ϱ, hence for any $\varepsilon>0$ there exists $\delta>0$ such that whenever $\varrho(x,y)<\delta$, then $\varrho(\varphi(x),\varphi(y))<\varepsilon$ for all $\varphi\in\mathcal{G}$ and in a consequence $d(x,y)\leqslant\varepsilon$. This finishes the proof. $\square$

We end this section with the following well-known Kakutani fixed point theorem.

Theorem 1.8. ([3]) *If K is a nonempty compact convex subset of a locally convex space and $\mathcal{F}$ is an equicontinuous group of affine automorphisms of K, then there exists a common fixed point for $\mathcal{F}$ in K.*

2. Groups

From now to the end of this section, X denotes a (nonempty) compact Hausdorff space and $\mathcal{G}$ any compact group of continuous transformations of X. It is enough to assume that $\mathcal{G}$ is an equicontinuous group (thanks to Corollaries 3.4 and 1.6 one can apply results of this section to the closure of $\mathcal{G}$).

Definition 2.1. *The quotient space* $X/\mathcal{G}$ is a topological space (with the quotient topology) generated by the following equivalence relation:

$$x \sim_{\mathcal{G}} y \iff \exists \varphi \in \mathcal{G}\colon\ \varphi(x) = y, \quad x, y \in X.$$

The classes of equivalence are just orbits of elements of X with respect to the group $\mathcal{G}$. In general (if X or $\mathcal{G}$ is not compact), the quotient space $X/\mathcal{G}$ does not need to be T_2. But in 'our' case it becomes true.

Proposition 2.2. *If $\mathcal{G}$ is a compact group of transformations of a compact Hausdorff space X, then the quotient space $X/\mathcal{G}$ is compact and T_2.*

Proof. Since X is compact and T_2, it suffices to show that the relation '$\sim_{\mathcal{G}}$' is closed as a subset of $X \times X$. But this relation is identical with the range of the mapping $X \times \mathcal{G} \ni (x, \varphi) \mapsto (x, \varphi(x)) \in X \times X$, which is continuous. So the range is compact and hence closed. □

Let $\pi_{\mathcal{G}} : X \to X/\mathcal{G}$ denote the projection map. It turns out that the space $\mathcal{C}(X/\mathcal{G}, \mathbb{R})$ can be identified with the algebra of those (continuous) functions from X to $\mathbb{R}$ which are constant on each class of equivalence. Such an identification is realized by means of the projection map.

Proposition 2.3. *If $\mathcal{A} := \{f \in \mathcal{C}(X, \mathbb{R}) \mid \forall \varphi \in \mathcal{G}\colon\ f \circ \varphi = f\}$, then the mapping $\Psi : \mathcal{C}(X/\mathcal{G}, \mathbb{R}) \ni f \mapsto f \circ \pi_{\mathcal{G}} \in \mathcal{A}$ is an isomorphism between* (*real*) *algebras.*

Proof. We only need to prove that Ψ is a bijection. Since the projection map is a surjection, Ψ is an injection. To prove that Ψ is a surjection, for any $g \in \mathcal{A}$ we define a function $f : X/\mathcal{G} \to \mathbb{R}$ by the formula:

$$f\big([x]_{\sim_{\mathcal{G}}}\big) := g(x), \quad x \in X.$$

Since g is constant on each class of equivalence, f is well defined. Moreover, $f \circ \pi_{\mathcal{G}} = g$, so f is continuous and $\Psi(f) = g$. □

The above proposition states that the quotient space is homeomorphic to the spectrum space of the algebra $\mathcal{A}$. The structure of the quotient space gives information about the structure of $\mathrm{Inv}(\mathcal{G})$, so it is useful to study it. Sometimes it can be easier to determine the algebra $\mathcal{A}$ than the orbits with respect to $\mathcal{G}$.

As an immediate consequence of Theorem 1.8 we obtain the following well-known result, the proof of which is given only for the sake of completeness.

Proposition 2.4. *If $\mathcal{G}$ is a compact group of continuous transformations of a* (*nonempty*) *compact Hausdorff space X, then the set* $\mathrm{Inv}(\mathcal{G})$ *is nonempty.*

Proof. Let $K := \mathrm{Prob}(X)$. K is a nonempty compact convex subset of a locally convex space. For any $\varphi \in \mathcal{G}$, let T_φ denote the following transformation of K:

$$T_\varphi(\mu) = \mu \circ \varphi, \quad \mu \in K.$$

Each T_φ is affine and continuous. Moreover, the set $\mathcal{H} := \{T_\varphi\colon\ \varphi \in \mathcal{G}\} \subset \mathcal{C}(K, K)$ is a group and the mapping $V : \mathcal{G} \ni \varphi \mapsto T_\varphi \in \mathcal{H}$ is an epimorphism of algebraical groups. What is more, V is continuous, which follows from Proposition 1.2 and Lemma 1.4. Therefore $\mathcal{H}$, as the range of V, is compact and the thesis of Theorem 1.8 finishes the proof. □

Now we are ready to state the main result of this section. This has been proved by Varadarajan in [7] in the metric case. The methods used by Varadarajan are different from ours and they seem not to work in general case.

Theorem 2.5. *If $\mathcal{G}$ is a compact group of continuous transformations of a compact Hausdorff space X, then the function $\Phi : \mathrm{Inv}(\mathcal{G}) \ni \mu \mapsto \mu \circ \pi_{\mathcal{G}} \in \mathrm{Prob}(X/\mathcal{G})$ is an affine homeomorphism of convex compact sets.*

Proof. It suffices to show that Φ is a bijection. To prove the surjectivity of Φ we take any $a \in X$ and consider its orbit $A := [a]_{\sim_{\mathcal{G}}} \neq \emptyset$. The set A is invariant under each mapping of $\mathcal{G}$, so the mapping $\Psi_a : \mathcal{G} \ni \varphi \mapsto \varphi|_A \in \mathcal{C}(A, A)$ is well defined. Moreover, it is a continuous homomorphism of semigroups with identities and therefore its range $\mathcal{G}_a := \Psi_a(\mathcal{G})$ is a compact group. Proposition 2.4 implies that there exists a measure $\lambda_a \in \mathrm{Inv}(\mathcal{G}_a)$ (defined on the Borel subsets of A!). Now if we put $\mu_a(B) := \lambda_a(B \cap A)$ for a Borel subset B of X, then $\mu_a \in \mathrm{Inv}(\mathcal{G})$ and $\mu_a \circ \pi_{\mathcal{G}}$ is Dirac's measure (on $X/\mathcal{G}$) at the point $[a]_{\sim_{\mathcal{G}}}$. So all Dirac's measures belongs to the range of Φ, which is compact and convex. Since the convex hull generated by all such measures is dense in $\mathrm{Prob}(X/\mathcal{G})$, this gives us the surjectivity.

To show that Φ is an injection, we take two different measures $\mu_1, \mu_2 \in \mathrm{Inv}(\mathcal{G})$. Since they are regular, there exists $f \in \mathcal{C}(X, \mathbb{R})$ such that $\int_X f \, d\mu_1 =: m_1 \neq m_2 := \int_X f \, d\mu_2$. Let $L_f : \mathcal{C}(X, X) \ni \psi \mapsto f \circ \psi \in \mathcal{C}(X, \mathbb{R})$ and $K_0 := L_f(\mathcal{G})$. X is compact, so L_f is continuous and K_0 is compact. Further, since $\mathcal{C}(X, \mathbb{R})$ with the compact-open topology can be considered as a Banach space (with the classical supremum norm), the closed convex hull $K := \operatorname{cl}\operatorname{conv} K_0$ of K_0 is also compact. By the continuity of the mapping (1.1) and thanks to the fact that $\mu_1, \mu_2 \in \mathrm{Inv}(\mathcal{G})$, we get that $\int_X h \, d\mu_j = m_j$ $(j = 1, 2)$ for any $h \in K_0$ and $h \in K$ too.

For $\varphi \in \mathcal{G}$, put

$$T_\varphi : K \ni h \longmapsto h \circ \varphi \in \mathcal{C}(X, \mathbb{R}).$$

Each T_φ is affine, continuous and injective and $T_\varphi(K_0) = K_0$, so $T_\varphi(K) = K$ $(\varphi \in \mathcal{G})$. Moreover, the set $\mathcal{T} := \{T_\varphi \colon \varphi \in \mathcal{G}\} \subset \mathcal{C}(K, K)$ is a group and the mapping $V : \mathcal{G} \ni \varphi \mapsto T_\varphi \in \mathcal{T}$ is a homomorphism of algebraical groups. It is also continuous, which follows from Proposition 1.2 and the previously mentioned fact that the operation of composing functions is continuous (in this case!), and hence $\mathcal{T}$ is compact. Now, by Theorem 1.8, there exists a common fixed point for $\mathcal{T}$, say $g \in K$. It means that $g \circ \varphi = g$ for any $\varphi \in \mathcal{G}$. Now Proposition 2.3 implies that there exists $u \in \mathcal{C}(X/\mathcal{G}, \mathbb{R})$ such that $u \circ \pi_{\mathcal{G}} = g$. Finally we have:

$$\int_{X/\mathcal{G}} u \, d(\mu_j \circ \pi_{\mathcal{G}}) = \int_X (u \circ \pi_{\mathcal{G}}) \, d\mu_j = \int_X g \, d\mu_j = m_j \quad (j = 1, 2)$$

which gives $\mu_1 \circ \pi_{\mathcal{G}} \neq \mu_2 \circ \pi_{\mathcal{G}}$ and so Φ is injective. □

Now one can easily prove the following generalization of the Haar measure theorem (for a generalization to a wider class of groups see [6]).

Corollary 2.6. *A compact group $\mathcal{G}$ has a unique invariant measure if and only if it is transitive, i.e. for any $x, y \in X$ there exists $\varphi \in \mathcal{G}$ such that $\varphi(x) = y$.*

Corollary 2.7. *If $X \neq \emptyset$ is a compact metric space which is transitive, i.e. for any $x, y \in X$ there exists an isometry*[1] *$\varphi : X \to X$ such that $\varphi(x) = y$, then there exists a unique Borel probabilistic measure*[2] *which is invariant under each isometry of X.*

Theorem 2.5 implies the fact that the set $\mathrm{Inv}(\mathcal{G})$ is a Choquet symplex. So it is enough to look for the extremal points of it. For this the following result can be useful.

[1] See the remark just before Lemma 3.1.

[2] Each finite Borel measure on a compact metric space is regular.

Corollary 2.8. *For a measure $\mu \in \mathrm{Inv}(\mathcal{G})$ the following conditions are equivalent*:

(i) *μ is an extremal point of* $\mathrm{Inv}(\mathcal{G})$,
(ii) *there exists $a \in X$ such that $\mu([a]_{\sim_\mathcal{G}}) = 1$.*

Proof. By Theorem 2.5, μ is an extremal point of $\mathrm{Inv}(\mathcal{G})$ if and only if $\mu \circ \pi_\mathcal{G}$ is an extremal point of $\mathrm{Prob}(X/\mathcal{G})$. But extremal points of the last mentioned set are exactly Dirac's measures. So μ is extremal if and only if $\mu(\pi_\mathcal{G}^{-1}([a]_{\sim_\mathcal{G}})) = 1$ for some $a \in X$, which finishes the proof. □

We end the section with the two classical examples.

Examples 2.9. (1) Let X and $\mathcal{G}$ be respectively the unit sphere in $\mathbb{R}^n$ and the group of all linear isometries restricted to X. By Corollary 2.6, the (probabilistic) Lebesgue measure on the manifold X is the unique one invariant under each isometry.

(2) Let X be the unit ball in $\mathbb{R}^n$ and again, let $\mathcal{G}$ stand for the group of all linear isometries restricted to X. It is easy to see that for any $x, y \in X$, $x \sim_\mathcal{G} y$ if and only if $\|x\| = \|y\|$ ($\|\cdot\|$ denotes the euclidean norm). Therefore the quotient space $X/\mathcal{G}$ is homeomorphic to the interval $[0, 1]$. Let m be the measure on X which is equal to the Lebesgue one on the unit sphere and is concentrated on it. Then the measures of the form $m \circ (tI)$, where $t \in [0, 1]$ and I is the identity map, are exactly the extremal points of $\mathrm{Inv}(\mathcal{G})$ and for each measure $\mu \in \mathrm{Inv}(\mathcal{G})$ there exists a unique one $\lambda \in \mathrm{Prob}([0, 1])$ such that $\mu = \int_0^1 m \circ (tI)\,\mathrm{d}\lambda(t)$.

3. Semigroups

In order to avoid misunderstandings, we need to establish the terminology. By an *isometry* between metric spaces (X, d) and (Y, ϱ) we mean a mapping $L : X \to Y$ which preserves the metrics, i.e. $\varrho(L(x'), L(x'')) = d(x', x'')$ for any $x', x'' \in X$. By such a definition, an isometry does not need to be a bijection, however

Lemma 3.1. [4] *If (X, d) is a compact metric space and $L : X \to X$ is an isometry, then it is a bijection.*

Now we are ready to state the main tool of this section.

Lemma 3.2. *If X is a compact Hausdorff space, $f : X \to X$ is a continuous mapping such that the semigroup $\mathcal{F}_0 := \{f^n\colon n \geqslant 1\}$ is equicontinuous $[f^n = f \circ \cdots \circ f]$ and $f(X) = X$, then f is a bijection and $f^{-1} \in \mathrm{cl}\,\mathcal{F}_0$.*

Proof. Since $\mathcal{F}_0$ is equicontinuous, $\mathcal{F} := \mathrm{cl}(\mathcal{F}_0 \cup \{\mathrm{id}_X\})$ is a compact semigroup. Let $x, y \in X$ and $x \neq y$. Let $g : X \to \mathbb{R}$ be a continuous mapping such that $g(x) \neq g(y)$. Put $T : \mathcal{C}(X, \mathbb{R}) \ni \varphi \mapsto \varphi \circ f \in \mathcal{C}(X, \mathbb{R})$. Since f is a surjection, T is an isometry on the Banach space $\mathcal{C}(X, \mathbb{R})$ (with the supremum norm). Let $K := \{g \circ h\colon h \in \mathcal{F}\}$. $\mathcal{F}$ is compact and so is K. Moreover, $T(K) \subset K$ (since $\mathcal{F}$ is a semigroup) and therefore, by Lemma 3.1, $T(K) = K$. It implies that there exists $h \in K$ such that $T(h) = g$, i.e. $h \circ f = g$. Now since $g(x) \neq g(y)$, also $f(x) \neq f(y)$ and so f is a bijection.

Now let $K_1, \ldots, K_r \subset X$ be compact, $U_1, \ldots, U_r \subset X$ be open and $f^{-1}(K_j) \subset U_j$ $(j = 1, \ldots, r)$. We need to prove that there exists $N \geqslant 1$ such that $f^N(K_j) \subset U_j$ $(j = 1, \ldots, r)$. Let $g_1, \ldots, g_r : X \to \mathbb{R}$ be such continuous mappings that $g_j|_{f^{-1}(K_j)} \equiv 0$ and $g_j|_{X \setminus U_j} \equiv 1$ $(j = 1, \ldots, r)$. Let $g := (g_1, \ldots, g_r) : X \to \mathbb{R}^r$ and $V : \mathcal{C}(X, \mathbb{R}^r) \ni \psi \mapsto \psi \circ f \in \mathcal{C}(X, \mathbb{R}^r)$. As in the previous part of the proof, V is an isometry (with respect to the supremum norm), since f is a surjection. Let $L_0 := \{g \circ f^n\colon n \geqslant 1\}$. Since $\mathcal{F}_0$ is equicontinuous, the closure of L_0, say L, is compact. Moreover, $T(L) \subset L$ and so $T(L) = L$. Since f is a bijection, so is T and therefore $T^{-2}(L) = L$. This implies that $g \circ f^{-1} \in L$, which means that there exists $N \geqslant 1$ such that $\sup_{x \in X} \|(g \circ f^N)(x) - (g \circ f^{-1})(x)\| < 1$. Since $(g_j \circ f^{-1})|_{K_j} \equiv 0$ and $g_j|_{X \setminus U_j} \equiv 1$, it is necessarily to be $f^N(K_j) \subset U_j$ $(j = 1, \ldots, r)$. □

Corollary 3.3. *If X is a compact Hausdorff space, $f : X \to X$ is a continuous mapping such that the set $\mathcal{F}_0 := \{f^n\colon n \geqslant 1\}$ is equicontinuous and $A \subset X$ is such a closed set that $f(A) \supset A$, then $f(A) = A$ and $f|_A : A \to A$ is a bijection.*

Proof. By the previous result, it is enough to prove that $f(A) = A$. Let $B := \mathrm{cl}(\bigcup_{n=0}^{\infty} f^n(A))$. Then $f(A) \subset B$ and $f(B) = B$, which follows from the inclusion $f(A) \supset A$. Let $g : B \to B$ be the restriction of f to the set B and $\mathcal{G}_0 := \{g^n\colon\ n \geqslant 1\} \subset \mathcal{C}(B, B)$. Since the operation of restricting mappings is continuous (in the compact-open topologies) and the closure of $\mathcal{F}_0$ is compact, so is the closure of $\mathcal{G}_0$. Hence the family $\mathcal{G}_0$ is equicontinuous. So, by the previous lemma, the mapping g is a bijection and $g^{-1} \in \mathrm{cl}\,\mathcal{G}_0$. Applying the same result to each function g^n $(n \geqslant 1)$ we obtain the inclusion $\{g^{-n}\colon\ n \geqslant 1\} \subset \mathrm{cl}\,\mathcal{G}_0$. So the set $\{g^{-n}\colon\ n \geqslant 1\}$ is equicontinuous. Moreover, $g^{-1}(A) \subset A$ (because $B \supset f(A) \supset A$). We have to show that $g^{-1}(A) = A$. Assume that $g^{-1}(A) \subsetneq A$. Take $a \in A \setminus g^{-1}(A)$ and a function $h \in \mathcal{C}(B, \mathbb{R})$ such that $h|_{g^{-1}(A)} \equiv 0$ and $h(a) = 1$. Let $K := \mathrm{cl}\{h \circ g^{-n}\colon\ n \geqslant 0\} \subset \mathcal{C}(B, \mathbb{R})$. K is compact and $T(K) \subset K$, where $T(v) = v \circ g^{-1}$ $(v \in \mathcal{C}(B, \mathbb{R}))$. Since g is a bijection, T is an isometry and therefore $T(K) = K \ni h$. It implies that there exists $N \geqslant 1$ such that $|h(x) - h(g^{-N}(x))| < 1$ for any $x \in B$. But for $x = a$ we have $h(a) = 1$ and $g^{-N}(a) \in g^{-1}(A)$, so $h(g^{-1}(a)) = 0$. This contradiction finishes the proof. □

Corollary 3.4. *If $\mathcal{F}_0$ is a nonempty equicontinuous semigroup of surjections from X onto X, then the closure of it is a compact group of homeomorphisms.*

Proof. Let $\mathcal{F}$ denote the closure of $\mathcal{F}_0$. Since the operation of composing functions is continuous and the set of all continuous surjections is closed in $\mathcal{C}(X, X)$, $\mathcal{F}$ is also a semigroup of surjections. By Lemma 3.2, $\mathcal{F}$ consists of homeomorphisms and is closed under the operation of taking inverses. Now nonemptiness of $\mathcal{F}_0$ gives $\mathrm{id}_X \in \mathcal{F}$ and this finishes the proof. □

Definition 3.5. *The fully invariant hull $\mathfrak{X} = \mathfrak{X}(\mathcal{F})$ corresponding to a family $\mathcal{F} \subset \mathcal{C}(X, X)$* is defined as the maximal set among all sets $A \subset X$ satisfying $f(A) = A$ for all $f \in \mathcal{F}$. In other words:

$$\mathfrak{X}(\mathcal{F}) := \bigcup \bigl\{A \subset X \mid \forall f \in \mathcal{F}\colon\ f(A) = A\bigr\}.$$

The fully invariant hull is always closed but it can be empty. It turns out that

Theorem 3.6. *If $\mathcal{G}$ is an equicontinuous semigroup, then the set* $\mathrm{Inv}(\mathcal{G})$ *is nonempty if and only if so is the set $\mathfrak{X} = \mathfrak{X}(\mathcal{G})$. If $\mathcal{G}$ and $\mathfrak{X}$ are nonempty, the family $\mathcal{F} := \mathrm{cl}(\mathcal{G}|_{\mathfrak{X}}) \subset \mathcal{C}(\mathfrak{X}, \mathfrak{X})$ is a compact group of continuous transformations of the space $\mathfrak{X}$ and the mapping $\Phi : \mathrm{Inv}(\mathcal{G}) \to \mathrm{Prob}(\mathfrak{X}/\mathcal{F})$ defined by the formula*

$$\bigl(\Phi(\mu)\bigr)(B) := \mu\bigl(\pi_{\mathcal{F}}^{-1}(B)\bigr), \quad B \in \mathfrak{B}(\mathfrak{X}/\mathcal{F}),^{3}\ \mu \in \mathrm{Inv}(\mathcal{G})$$

[*for short*: $\Phi(\mu) = \mu \circ \pi_{\mathcal{F}}$] *is an affine homeomorphism between compact convex sets.*

Proof. First of all we have to prove that if $\mu \in \mathrm{Inv}(\mathcal{G})$, then $\mathrm{supp}\,\mu \subset \mathfrak{X}$, where $\mathrm{supp}\,\mu := \bigcap\{A \subset X\colon\ A = \mathrm{cl}\,A,\ \mu(A) = 1\}$. Let $\mu \in \mathrm{Inv}(\mathcal{G})$ and $f \in \mathcal{G}$. Then $\mu(f(\mathrm{supp}\,\mu)) = (\mu \circ f)(f(\mathrm{supp}\,\mu)) = \mu(f^{-1}(f(\mathrm{supp}\,\mu))) \geqslant \mu(\mathrm{supp}\,\mu) = 1$, so $\mathrm{supp}\,\mu \subset f(\mathrm{supp}\,\mu)$ and by Corollary 3.3, $f(\mathrm{supp}\,\mu) = \mathrm{supp}\,\mu$. Therefore $\mathfrak{X}$, as the union of such sets, includes $\mathrm{supp}\,\mu$. It means that if $\mathrm{Inv}(\mathcal{G}) \neq \emptyset$, then $\mathfrak{X} \neq \emptyset$ (because $\mathrm{supp}\,\mu$ is always nonempty).

Now if we assume that $\mathfrak{X} \neq \emptyset$, then the family $\mathcal{F}$ is a nonempty equicontinuous semigroup of surjections from $\mathfrak{X}$ onto $\mathfrak{X}$ and so, by Corollary 3.4, it is a compact group of homeomorphisms. Further, by Theorem 2.5, the set $\mathrm{Inv}(\mathcal{F})$ is nonempty and the mapping $\mathrm{Inv}(\mathcal{F}) \ni \mu \mapsto \mu \circ \pi_{\mathcal{F}} \in \mathrm{Prob}(\mathfrak{X}/\mathcal{F})$ is an isomorphism. Now the thesis follows from the fact that the mapping $M : \mathrm{Inv}(\mathcal{G}) \ni \mu \mapsto \mu|_{\mathfrak{B}(\mathfrak{X})} \in \mathrm{Inv}(\mathcal{F})$ is a (well defined) isomorphism (the inverse map is of the form $(M^{-1}(\mu))(B) = \mu(B \cap \mathfrak{X})$), since for any $\mu \in \mathrm{Inv}(\mathcal{G})$ we have $\mathrm{supp}\,\mu \subset \mathfrak{X}$. □

The above theorem reduces the issue of the nonemptiness of the set $\mathrm{Inv}(\mathcal{G})$ to the problem of the existence of a nonempty set $A \subset X$ such that $f(A) = A$ for any $f \in \mathcal{G}$.

Definition 3.7. *The* (*first order*) *common range of a family $\mathcal{F} \subset \mathcal{C}(X, X)$* is the set

$$\mathcal{R}(\mathcal{F}) = \mathcal{R}^1(\mathcal{F}) := \bigcap_{f \in \mathcal{F} \cup \{\mathrm{id}_X\}} f(X).$$

[3] $\mathfrak{B}(U)$ stands for the σ-algebra of all Borel subsets of a topological space U.

More general, *the nth order common range of* $\mathcal{F}$ is defined by the inductive formula:

$$\mathcal{R}^{n+1}(\mathcal{F}) := \bigcap_{f\in\mathcal{F}\cup\{\mathrm{id}_X\}} f\big(\mathcal{R}^n(\mathcal{F})\big).$$

Finally, the set $\mathcal{R}^\infty(\mathcal{F}) := \bigcap_{n=1}^\infty \mathcal{R}^n(\mathcal{F})$ is called *the infinite order common range of* $\mathcal{F}$. By the definition, the sequence $(\mathcal{R}^n(\mathcal{F}))_{n=1}^\infty$ is decreasing.

Proposition 3.8. (1) *If* $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{C}(X,X)$, *then* $\mathcal{R}^n(\mathcal{F}_1) \supset \mathcal{R}^n(\mathcal{F}_2)$ *for any* $n \in \mathbb{N}_* \cup \{\infty\}$.

(2) *If* $\mathcal{F}$ *is a nonempty family of continuous transformations of a compact Hausdorff space* X, *then for any* $n \in \mathbb{N}_* \cup \{\infty\}$:

$$\mathcal{R}^n(\mathcal{F}) = \bigcap_{\substack{f_1,\dots,f_r\in\mathcal{F}\\ r\geqslant 1}} \mathcal{R}^n\big(\{f_1,\dots,f_r\}\big).$$

Proof. (1) Clear.

(2) The inclusion "$\subset$" follows from (1). To prove the other one, we argue by the induction. For $n = 1$ the equality is immediate. Now, if we assume that it is satisfied for n, then for any $f \in \mathcal{F}\cup\{\mathrm{id}_X\}$ the following formula holds (since the family $\{\mathcal{R}^n(\{f_1,\dots,f_r\})\}_{f_1,\dots,f_r\in\mathcal{F},\ r\geqslant 1}$ is directed by the relation '$\supset$' and consists of compact sets):

$$f\Bigg(\bigcap_{\substack{f_1,\dots,f_r\in\mathcal{F}\\ r\geqslant 1}} \mathcal{R}^n\big(\{f_1,\dots,f_r\}\big)\Bigg) = \bigcap_{\substack{f_1,\dots,f_r\in\mathcal{F}\\ r\geqslant 1}} f\big(\mathcal{R}^n\big(\{f_1,\dots,f_r\}\big)\big).$$

So, thanks to (1) and the induction hypothesis, for any $g \in \mathcal{F}$ we have:

$$\begin{aligned}
&\bigcap_{\substack{f_1,\dots,f_r\in\mathcal{F}\\ r\geqslant 1}} \mathcal{R}^{n+1}\big(\{f_1,\dots,f_r\}\big) \subset \bigcap_{\substack{f_1,\dots,f_r\in\mathcal{F}\\ r\geqslant 2,\ f_1=g}} \mathcal{R}^{n+1}\big(\{f_1,\dots,f_r\}\big)\\
&\quad= \bigcap_{\substack{f_1,\dots,f_r\in\mathcal{F}\\ r\geqslant 1}} \mathcal{R}^{n+1}\big(\{g,f_1,\dots,f_r\}\big) \subset \bigcap_{\substack{f_1,\dots,f_r\in\mathcal{F}\\ r\geqslant 1}} g\big(\mathcal{R}^n\big(\{g,f_1,\dots,f_r\}\big)\big)\\
&\quad\subset \bigcap_{\substack{f_1,\dots,f_r\in\mathcal{F}\\ r\geqslant 1}} g\big(\mathcal{R}^n\big(\{f_1,\dots,f_r\}\big)\big)\\
&\quad= g\Bigg(\bigcap_{\substack{f_1,\dots,f_r\in\mathcal{F}\\ r\geqslant 1}} \mathcal{R}^n\big(\{f_1,\dots,f_r\}\big)\Bigg) = g\big(\mathcal{R}^n(\mathcal{F})\big).
\end{aligned}$$

Moreover:

$$\bigcap_{\substack{f_1,\dots,f_r\in\mathcal{F}\\ r\geqslant 1}} \mathcal{R}^{n+1}\big(\{f_1,\dots,f_r\}\big) \subset \bigcap_{\substack{f_1,\dots,f_r\in\mathcal{F}\\ r\geqslant 1}} \mathcal{R}^n\big(\{f_1,\dots,f_r\}\big) = \mathcal{R}^n(\mathcal{F}) = \mathrm{id}_X\big(\mathcal{R}^n(\mathcal{F})\big).$$

Now it suffices to take the intersection of all the sets $g(\mathcal{R}^n(\mathcal{F}))$, over all $g \in \mathcal{F}\cup\{\mathrm{id}_X\}$, to get the needed inclusion for $n+1$. The induction argument gives us the required equality for each finite n. Finally we have:

$$\begin{aligned}
\mathcal{R}^\infty(\mathcal{F}) &= \bigcap_{n=1}^\infty \mathcal{R}^n(\mathcal{F}) = \bigcap_{n=1}^\infty \bigcap_{\substack{f_1,\dots,f_r\in\mathcal{F}\\ r\geqslant 1}} \mathcal{R}^n\big(\{f_1,\dots,f_r\}\big)\\
&= \bigcap_{\substack{f_1,\dots,f_r\in\mathcal{F}\\ r\geqslant 1}} \bigcap_{n=1}^\infty \mathcal{R}^n\big(\{f_1,\dots,f_r\}\big) = \bigcap_{\substack{f_1,\dots,f_r\in\mathcal{F}\\ r\geqslant 1}} \mathcal{R}^\infty\big(\{f_1,\dots,f_r\}\big). \qquad \square
\end{aligned}$$

Proposition 3.9. *If $\mathcal{G}$ is an equicontinuous semigroup of transformations of a compact Hausdorff space X, then $\mathcal{X}(\mathcal{G}) = \mathcal{R}^\infty(\mathcal{G})$. Moreover, $\mathcal{X}(\mathcal{G}) \neq \emptyset$ if and only if $\mathcal{R}^n(\mathcal{G}) \neq \emptyset$ for all $n \geqslant 1$.*

Proof. Let $B := \mathcal{X}(\mathcal{G})$. We know that $f(B) = B$ for any $f \in \mathcal{G}$. Since $\mathcal{R}^n(\mathcal{G}) \supset \mathcal{R}^{n+1}(\mathcal{G})$ $(n \geqslant 1)$ and all common ranges are compact, we have $f(\mathcal{R}^\infty(\mathcal{G})) = \bigcap_{n=1}^\infty f(\mathcal{R}^n(\mathcal{G})) \supset \bigcap_{n=1}^\infty \mathcal{R}^{n+1}(\mathcal{G}) = \mathcal{R}^\infty(\mathcal{G})$ $(f \in \mathcal{G})$. Hence, by Corollary 3.3, $f(\mathcal{R}^\infty(\mathcal{G})) = \mathcal{R}^\infty(\mathcal{G})$ for any $f \in \mathcal{G}$ and so $\mathcal{R}^\infty(\mathcal{G}) \subset B$. On the other hand, if $f \in \mathcal{G}$, then $B = f(B) \subset f(X)$ and $B \subset \bigcap_{f \in \mathcal{G} \cup \{\mathrm{id}_X\}} f(X) = \mathcal{R}^1(\mathcal{G})$. Now if $B \subset \mathcal{R}^n(\mathcal{G})$ for some $n \geqslant 1$, then $B = f(B) \subset f(\mathcal{R}^n(\mathcal{G}))$ $(f \in \mathcal{G})$ and hence $B \subset \bigcap_{f \in \mathcal{G} \cup \{\mathrm{id}_X\}} f(\mathcal{R}^n(\mathcal{G})) = \mathcal{R}^{n+1}(\mathcal{G})$. The induction argument gives us $B \subset \mathcal{R}^n(\mathcal{G})$ for all $n \geqslant 1$ and finally $B \subset \bigcap_{n=1}^\infty \mathcal{R}^n(\mathcal{G}) = \mathcal{R}^\infty(\mathcal{G})$. To end the proof it is enough to observe that $\mathcal{R}^\infty(\mathcal{G}) \neq \emptyset$ if and only if $\mathcal{R}^n(\mathcal{G}) \neq \emptyset$ for all $n \geqslant 1$, which follows from the fact that the sequence $(\mathcal{R}^n(\mathcal{G}))_{n=1}^\infty$ is decreasing and consists of compact sets. □

To end the paper we give a few conditions equivalent to the nonemptiness of the set $\mathrm{Inv}(\mathcal{G})$.

Proposition 3.10. *Let $\mathcal{F} \subset \mathcal{C}(X, X)$ be such a family that the semigroup $\mathcal{G}$ generated by it is equicontinuous. Then the following conditions are equivalent*:

(i) $\mathrm{Inv}(\mathcal{F}) \neq \emptyset$,
(ii) *for any $f_1, \dots, f_r \in \mathcal{F}$ $(r \geqslant 1)$ and each $n \geqslant 1$ the set $\mathcal{R}^n(\{f_1, \dots, f_r\})$ is nonempty,*
(iii) *there exists $a \in X$ such that the set $A := \mathrm{cl}\{g(a)\colon\ g \in \mathcal{G}\}$ is fully invariant for $\mathcal{F}$, i.e. $f(A) = A$ for any $f \in \mathcal{F}$.*

Proof. All the equivalences are easily seen to hold true for $\mathcal{F} = \emptyset$. So we may assume that the family is nonempty.

(i) ⇒ (ii): This follows immediately from Corollary 1.6, Propositions 3.9 and 3.8(1).

(ii) ⇒ (i): By Proposition 3.9(2), $\mathcal{R}^\infty(\mathcal{F}) \neq \emptyset$ (since the corresponding family $\{\mathcal{R}^\infty(\{f_1, \dots, f_r\})\}_{f_1,\dots,f_r \in \mathcal{F},\ r \geqslant 1}$ is directed by the relation '$\supset$' and consists of nonempty compact sets). The same argument as in the proof of Proposition 3.9 shows that $f(\mathcal{R}^\infty(\mathcal{F})) = \mathcal{R}^\infty(\mathcal{F})$ for every $f \in \mathcal{F}$. Since each element of $\mathcal{G}$ is a composition of a finite number of functions from $\mathcal{F}$, the same equality holds also for every $f \in \mathcal{G}$. It means that $\mathcal{X}(\mathcal{G}) \neq \emptyset$ and, by Theorem 3.6, $\mathrm{Inv}(\mathcal{G}) \neq \emptyset$.

(i) ⇒ (iii): If $\mathrm{Inv}(\mathcal{F}) \neq \emptyset$, then also $\mathrm{Inv}(\mathcal{G}) \neq \emptyset$ and $\mathcal{X}(\mathcal{G}) \neq \emptyset$. Now by Theorems 3.6 and 2.5, each element of $\mathcal{X}(\mathcal{G})$ satisfies the condition in (iii).

(iii) ⇒ (i): Observe that $\mathcal{X}(\mathcal{G}) \neq \emptyset$ and apply Theorem 3.6. □

Example 3.11. The following example shows that the nonemptiness of the first common range of a compact semigroup does not imply the nonemptiness of the set of invariant measures.

Let $X := [0, 1]$ and $f : X \ni x \mapsto |\frac{1}{2} - x| \in X$. It is easy to check that the family $\mathcal{F} := \{\mathrm{id}_X, f, 1 - f, \frac{1}{2} - f, \frac{1}{2} + f\}$ is a finite semigroup of continuous transformations of the space X (so $\mathcal{F}$ is compact) and $\mathcal{R}^1(\mathcal{F}) = \{\frac{1}{2}\}$ but $\mathcal{R}^2(\mathcal{F}) = \emptyset$ and therefore $\mathrm{Inv}(\mathcal{F})$ is empty.

As a consequence of Theorem 3.6 we get the following result (a necessary condition for the existence of an invariant measure is stated in [2, Proposition 2.7] in more general case).

Theorem 3.12. *If $\mathcal{G}$ is a compact* (*Hausdorff*) *semigroup, then the following conditions are equivalent*:

(i) *there exists an invariant* (*regular, Borel and probabilistic*) *measure on $\mathcal{G}$,*
(ii) *there exists $e \in \mathcal{G}$ such that $e^2 = e$, the set $\mathcal{G} \cdot e$ is a group*[4] *and $xe = ex$ for any $x \in \mathcal{G}$.*

Moreover, if (i) *holds, the invariant measure is unique. The element e in the condition* (ii) *is also uniquely determined and it satisfies the following condition*:

$$\forall a \in \mathcal{G}\colon\ (a^2 = a \Rightarrow ae = ea = e).$$

[4] If $\mathcal{G}$ is not a group, the unit of the group $\mathcal{G} \cdot e$ *is not* the unit of $\mathcal{G}$, even if $\mathcal{G}$ has the neutral element.

Proof. We may assume that $\mathcal{G} \neq \emptyset$. For $a, b \in \mathcal{G}$, let $L_a(x) := ax$, $R_b(x) := xb$ $(x \in \mathcal{G})$ and $B_{a,b} := L_a \circ R_b$. Then the set $\mathcal{B} := \{L_a, B_{a,b}, R_b\colon\ a, b \in \mathcal{G}\}$ is a semigroup of continuous transformations of the space $\mathcal{G}$. Moreover, it is compact (since the multiplication is continuous). So, by Theorem 3.6, there exists an invariant measure if and only if $\mathcal{X}(\mathcal{B}) \neq \emptyset$.

(ii) $\Rightarrow$ (i): Let $A := \mathcal{G} \cdot e = e \cdot \mathcal{G} \neq \emptyset$. It suffices to show that A is fully invariant for $\mathcal{B}$. Let $a, b \in \mathcal{G}$ and $x \in A$. Since $A = \mathcal{G} \cdot e$, $x = x_1 e$ for some $x_1 \in \mathcal{G}$ and then $B_{a,b}(x) = axb = ax_1eb = (ax_1b)e \in \mathcal{G} \cdot e = A$, so $B_{a,b}(A) \subset A$. On the other hand, A is a group and $e \in A$ is an idempotent, so e is the unit of A. Therefore for $a' := ae \in A$ and $b' := eb \in A$ there exist $c, d \in A$ such that $a'c = e = db'$. And hence $cxd \in A$ and $B_{a,b}(cxd) = acxdb = a(ec)x(de)b = a'cxdb' = exe = x$, which gives $B_{a,b}(A) \supset A$. Finally $L_a(A) = B_{a,e}(A) = A$ and similarly $R_b(A) = B_{e,b}(A) = A$.

(i) $\Rightarrow$ (ii): Let $H := \mathcal{X}(\mathcal{B})$. Since $\mathrm{Inv}(\mathcal{B}) \neq \emptyset$, $H \neq \emptyset$ too. By Theorem 3.6, the family $\mathcal{B}|_H$ consists of homeomorphisms of the space H, which means that H is a subsemigroup of $\mathcal{G}$ and all mappings $L_a|_H$ and $R_a|_H$ $(a \in H)$ are permutations of H. So, by Lemma 3.13 (see further), H is a group. Let $e \in H$ be the unit of H. Clearly $e^2 = e$. We shall prove that $\mathcal{G} \cdot e = e \cdot \mathcal{G} = H$ and that e commutes with all elements of $\mathcal{G}$. On the one hand, $H = H \cdot e \subset \mathcal{G} \cdot e$ and analogously $H \subset e \cdot \mathcal{G}$, but on the other hand, for $a \in \mathcal{G}$, $ae = L_a(e) \in L_a(H) = H$, $ea = R_a(e) \in R_a(H) = H$ and $ea = (ea)e = e(ae) = ae$, which finishes the proof of this implication.

It remains to prove the last part of the theorem. Firstly, the proof of the implication '(i) $\Rightarrow$ (ii)' shows that if an invariant measure exists, the set $H = \mathcal{X}(\mathcal{B})$ is a group, so the group $\mathcal{B}|_H$ is transitive and therefore the set $\mathrm{Inv}(\mathcal{B}|_H)$ is a singleton and so is $\mathrm{Inv}(\mathcal{B})$. Secondly, the proof of the inverse implication shows that if $e' \in \mathcal{G}$ satisfies the condition (ii) of the theorem, then $e' \in H$. But H is a group and it has only one idempotent, which gives the uniqueness of e. Finally, if $a \in \mathcal{G}$ is an idempotent, then $ae \in H$, $ae = a(ae) = a(eae) = (ae)^2$ and hence $ae = e$ (since $ae \in H$ is an idempotent). $\square$

In the proof of the above theorem we used the following algebraical result, the proof of which we leave to the Reader as an exercise.

Lemma 3.13. *If G is a **nonempty** semigroup such that all mappings L_a and R_a $(a \in G)$ are bijections, then G is a group.*

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